

# Shadow price of information in discrete time stochastic optimization

Teemu Pennanen\*      Ari-Pekka Perkkiö†

January 21, 2016

**Dedicated to R. T. Rockafellar on his 80th Birthday**

## Abstract

The shadow price of information has played a central role in stochastic optimization ever since its introduction by Rockafellar and Wets in the mid-seventies. This article studies the concept in an extended formulation of the problem and gives relaxed sufficient conditions for its existence. We allow for general adapted decision strategies, which enables one to establish the existence of solutions and the absence of a duality gap e.g. in various problems of financial mathematics where the usual boundedness assumptions fail. As applications, we calculate conjugates and subdifferentials of integral functionals and conditional expectations of normal integrands. We also give a dual form of the general dynamic programming recursion that characterizes shadow prices of information.

## 1 Introduction

Let  $(\Omega, \mathcal{F}, P)$  be a probability space with a filtration  $(\mathcal{F}_t)_{t=0}^T$  and consider the multistage stochastic optimization problem

$$\text{minimize } Eh(x) \quad \text{over } x \in \mathcal{N}, \quad (\text{SP})$$

where  $\mathcal{N} = \{(x_t)_{t=0}^T \mid x_t \in L^0(\Omega, \mathcal{F}_t, P; \mathbb{R}^{n_t})\}$  denotes the space of decision strategies adapted to the filtration,  $h$  is a convex normal integrand on  $\mathbb{R}^n \times \Omega$  and  $Eh$  denotes the associated integral functional on  $L^0(\Omega, \mathcal{F}, P; \mathbb{R}^n)$ . Here and in what follows,  $n = n_0 + \dots + n_T$  and  $L^0(\Omega, \mathcal{F}, P; \mathbb{R}^n)$  denotes the linear space of equivalence classes of  $\mathbb{R}^n$ -valued  $\mathcal{F}$ -measurable functions. As usual, two functions are equivalent if they are equal  $P$ -almost surely. Throughout, we

---

\*Department of Mathematics, King's College London, Strand, London, WC2R 2LS, United Kingdom

†Department of Mathematics, Technische Universität Berlin, Building MA, Str. des 17. Juni 136, 10623 Berlin, Germany. The author is grateful to the Einstein Foundation for the financial support.

define the expectation of a measurable function as  $+\infty$  unless its positive part is integrable.

Problems of the form (SP) have been extensively studied since their introduction in the mid 70's; see [16, 17, 19]. Despite its simple appearance, problem (SP) is a very general format of stochastic optimization. Indeed, various pointwise (almost sure) constraints can be incorporated in the objective by assigning  $f$  the value  $+\infty$  when the constraints are violated. Several examples can be found in the above references. Applications to financial mathematics are given in [8, 10, 9]. Our formulation of problem (SP) extends its original formulations by allowing for general filtrations  $(\mathcal{F}_t)_{t=0}^T$  as well as general adapted strategies instead of bounded ones. This somewhat technical extension turns out to be quite convenient e.g. in financial applications.

We will use the short hand notation  $L^\infty := L^\infty(\Omega, \mathcal{F}, P; \mathbb{R}^n)$  and define the function  $\phi : L^\infty \rightarrow \overline{\mathbb{R}}$  by

$$\phi(z) = \inf_{x \in \mathcal{N}} Eh(x + z).$$

We assume throughout that  $\phi(0)$  is finite and that  $Eh$  is proper on  $L^\infty$ . Clearly  $\phi(0)$  is the optimum value of (SP) while in general,  $\phi(z)$  gives the optimum value that can be achieved in combination with an essentially bounded *nonadapted* strategy  $z$ . Note also that  $\phi(z) = \phi(0)$  for all  $z \in L^\infty \cap \mathcal{N}$ .

The space  $L^\infty$  is in separating duality with  $L^1 := L^1(\Omega, \mathcal{F}, P; \mathbb{R}^n)$  under the bilinear form

$$\langle z, v \rangle := E(z \cdot v).$$

A  $v \in L^1$  is said to be a *shadow price of information* for problem (SP) if it is a subgradient of  $\phi$  at the origin, i.e., if

$$\phi(z) \geq \phi(0) + \langle z, v \rangle \quad \forall z \in L^\infty.$$

The following result, the proof of which is given in the appendix, shows that the shadow price of information has the same fundamental properties here as in Rockafellar and Wets [19] where the primal solutions were restricted to be essentially bounded. Here and in what follows,  $\phi^*$  denotes the *conjugate* of  $\phi$  defined for each  $v \in L^1$  as

$$\phi^*(v) = \sup_{z \in L^\infty} \{\langle z, v \rangle - \phi(z)\}.$$

The *annihilator* of  $\mathcal{N}^\infty$  will be denoted by  $\mathcal{N}^\perp := \{v \in L^1 \mid \langle z, v \rangle = 0 \forall z \in \mathcal{N}^\infty\}$ .

**Theorem 1.** *We have  $\phi^* = Eh^* + \delta_{\mathcal{N}^\perp}$ . In particular,  $v \in L^1$  is a shadow price of information if and only if it solves the dual problem*

$$\text{minimize} \quad Eh^*(v) \quad \text{over } v \in \mathcal{N}^\perp$$

*and the optimum value equals  $-\phi(0)$ . In this case, an  $x \in \mathcal{N}$  is optimal if and only if  $Eh(x) < 0$  and it minimizes the function  $x \mapsto h(x, \omega) - v(\omega) \cdot x$  almost surely.*

The notion of a shadow price of information first appeared in a general single period model in Rockafellar [15, Example 6 in Section 10] and Rockafellar and Wets [18, Section 4]. Extension to finite discrete time was given in [19]. Continuous-time extensions have been studied in Wets [24], Back and Pliska [1], Davis [4] and Davis and Burstein [5] under various structural assumptions. The shadow price of information has been found useful in formulating dual problems and deriving optimality condition in general parametric stochastic optimization problems; see e.g. [20, 1, 2]. The shadow price of information is useful also in subdifferential calculus involving conditional expectations; see [21] and Section 3.2 below. As a further application, we give a dual formulation of the general dynamic programming recursion from [19] and [6]; see Section 3.3.

The main result of this paper gives new generalized sufficient conditions for the existence of a shadow price of information for the discrete time problem (SP). Its proof is obtained by extending the original argument of [19] and by relaxing some of the technical assumptions made there. As already noted, we do not require the decision strategies to be essentially bounded. This allows one to establish the existence of solutions and the absence of a duality gap e.g. in various problems in financial mathematics; see [10, 11]. We also relax the assumptions made in [19] on the normal integrand  $h$ .

We will denote the *adapted projection* of an  $x \in L^\infty$  by  ${}^a x$ , that is,  $({}^a x)_t = E_t x_t$ , where  $E_t$  denotes the conditional expectation with respect to  $\mathcal{F}_t$ . We will also use the notation  $x^t := (x_0, \dots, x_t)$ .

**Assumption 1.** For every  $z \in \text{dom } Eh \cap L^\infty$  and every  $t = 0, \dots, T$ , there exists  $\hat{z} \in \text{dom } Eh \cap L^\infty$  such that  $E_t z^t = \hat{z}^t$ .

It was assumed in [19] (conditions C and D, respectively) that the sets  $\text{dom } h(\cdot, \omega)$  are closed, uniformly bounded, and “nonanticipative” and that there exists a  $\mu \in L^1$  such that  $|h(x, \omega)| \leq \mu(\omega)$  for all  $x \in \text{dom } h(\cdot, \omega)$ . The nonanticipativity means the projection mappings  $D^t(\omega) := \{x^t \mid x \in \text{dom } h(\cdot, \omega)\}$  are  $\mathcal{F}_t$ -measurable for all  $t$ . These conditions imply Assumption 1. Indeed, if  $z \in \text{dom } Eh \cap L^\infty$ , then  $z^t \in D^t(\omega)$  almost surely and, by Jensen’s inequality,  $E_t z^t \in \text{dom } h$  almost surely as well. By the measurable selection theorem (see [22, Corollary 14.6]), there exists a  $\hat{z} \in L^0$  such that  $\hat{z} \in \text{dom } h$  and  $\hat{z}^t = E_t z^t$  almost surely. The uniform boundedness of  $\text{dom } h$  implies that  $\hat{z} \in L^\infty$  while the upper bound  $\mu$  gives  $Eh(\hat{z}) < \infty$ .

We will also use the following.

**Assumption 2.** There exists  $\rho \in \mathbb{R}$  such that, for every  $z \in \text{aff dom } Eh \cap L^\infty$ , there exists  $x \in \text{aff dom } Eh \cap \mathcal{N}^\infty$  such that  $\|x - z\| \leq \rho \|{}^a z - z\|$ .

Assumption 2 holds, in particular, if  ${}^a z \in \text{aff dom } Eh$  for all  $z \in \text{aff dom } Eh \cap L^\infty$ . In the single-step case where  $T = 0$ , this latter condition coincides with Assumption 1. Assumption 2 is also implied by the *strict feasibility* assumption made in [19, Theorem 2]. Indeed, strict feasibility implies that  $\text{dom } Eh$  contains an open ball so that  $\text{aff dom } Eh = L^\infty$ .

In order to clarify the structure and the logic of its proof, we have split our main result in two statements of independent interest, Theorems 4 and 5 below. Combining them gives the following extension of [19, Theorem 2].

**Theorem 2.** *Let Assumption 1 and 2 hold, and assume that  $Eh$  is strongly continuous at a point of  $\mathcal{N}^\infty$  relative to  $\text{aff dom } Eh \cap L^\infty$ . Then a shadow price of information exists.*

A sufficient condition for the relative continuity will be given in Theorem 6 below. It is obtained by extending the argument in the proof of [14, Theorem 2].

## 2 Existence of a shadow price of information

Our main results are derived by analyzing the auxiliary value function  $\tilde{\phi} : L^\infty \rightarrow \mathbb{R}$  defined by

$$\tilde{\phi}(z) = \inf_{x \in \mathcal{N}^\infty} Eh(x + z).$$

Here decision strategies are restricted to be essentially bounded like in [19]. Clearly  $\tilde{\phi} \geq \phi$ . Our strategy is to establish the existence of a subgradient of  $\tilde{\phi}$  at the origin much like in [19]. By the following simple lemma, this will then serve as a shadow price of information for the general problem (SP). Following [13], we denote the *biconjugate* of a function  $f$  by  $\text{cl } f := f^{**}$ .

**Lemma 3.** *We have  $\text{cl } \tilde{\phi} = \text{cl } \phi$ . If  $\partial \tilde{\phi}(0)$  is nonempty, then  $\partial \tilde{\phi}(0) = \partial \phi(0)$ .*

*Proof.* By the interchange rule [22, Theorem 14.60] again,  $\tilde{\phi}^* = Eh^* + \delta_{\mathcal{N}^\perp}$  (see the proof of Theorem 1). Thus  $\tilde{\phi}^* = \phi^*$ , so  $\text{cl } \tilde{\phi} = \text{cl } \phi$ . If  $\partial \tilde{\phi}(0) \neq \emptyset$ , then  $\tilde{\phi}(0) = \text{cl } \tilde{\phi}(0)$  so  $\tilde{\phi}(0) = \phi(0)$  (since we always have  $\tilde{\phi} \geq \phi \geq \text{cl } \phi$ ), by the first part, so  $v \in \partial \tilde{\phi}(0)$  iff  $v \in \partial \phi(0)$ .  $\square$

The general idea in [19] was first to prove the existence of a subgradient for  $\tilde{\phi}$  with respect to the pairing of  $L^\infty$  with its Banach dual  $(L^\infty)^*$ . This was then modified to get a subgradient with respect to the pairing of  $L^\infty$  with  $L^1 \subset (L^\infty)^*$ . By [25], any  $v \in (L^\infty)^*$  can be expressed as  $v = v^a + v^s$  where  $v^a \in L^1$  and  $v^s \in (L^\infty)^*$  is such that there is a decreasing sequence of sets  $A^\nu \in \mathcal{F}$  such that  $P(A^\nu) \searrow 0$  and

$$\langle z, v^s \rangle = 0$$

for any  $z \in L^\infty$  that vanishes on  $A^\nu$ . The representation  $v = v^a + v^s$  is known as the *Yosida–Hewitt decomposition* of  $v$ . In order to control the singular component  $v^s$ , we have introduced Assumption 1.

Below, the *strong topology* will refer to the norm topology of  $L^\infty$ .

**Theorem 4.** *Let Assumption 1 hold. If  $\tilde{\phi}$  is proper and strongly closed at the origin, then  $\phi$  is closed at the origin and  $\phi(0) = (\text{cl } \tilde{\phi})(0)$ . If  $\tilde{\phi}$  is strongly subdifferentiable at the origin, then  $\partial \phi(0) = \partial \tilde{\phi}(0) \neq \emptyset$ .*

*Proof.* By Lemma 3, the first claim holds as soon as  $\tilde{\phi}(0) = \text{cl}\tilde{\phi}(0)$ , while the second holds if  $\partial\tilde{\phi}(0) \neq \emptyset$ . Strong closedness of  $\tilde{\phi}$  at the origin means that for every  $\epsilon > 0$  there is a  $v \in (L^\infty)^*$  such that  $\tilde{\phi}(0) \leq -\tilde{\phi}^*(v) + \epsilon$ , or equivalently,

$$\begin{aligned} \tilde{\phi}(z) &\geq \tilde{\phi}(0) + \langle z, v \rangle - \epsilon \quad \forall z \in L^\infty \\ \iff Eh(x+z) &\geq \tilde{\phi}(0) + \langle z, v \rangle - \epsilon \quad \forall z \in L^\infty, x \in \mathcal{N}^\infty \\ \iff Eh(z) &\geq \tilde{\phi}(0) + \langle z-x, v \rangle - \epsilon \quad \forall z \in L^\infty, x \in \mathcal{N}^\infty, \end{aligned}$$

which means that  $v \perp \mathcal{N}^\infty$  and

$$Eh(z) \geq \tilde{\phi}(0) + \langle z, v \rangle - \epsilon \quad \forall z \in L^\infty. \quad (1)$$

Similarly,  $\tilde{\phi}$  is strongly subdifferentiable at the origin iff  $v \perp \mathcal{N}^\infty$  and (1) holds with  $\epsilon = 0$ .

We will prove the existence of a  $v \perp \mathcal{N}^\infty$  which has  $v^s = 0$  and satisfies (1) with  $\epsilon$  multiplied with  $2^{T+1}$ . Similarly to the above, this means that  $\phi$  is closed (if (1) holds with all  $\epsilon > 0$ ) or subdifferentiable (if  $\epsilon = 0$ ) at the origin with respect to the weak topology. The existence will be proved recursively by showing that if  $v \perp \mathcal{N}^\infty$  satisfies (1) and  $v_s^s = 0$  for  $s > t$  (this holds for  $t = T$  as noted above), then there exists a  $\tilde{v} \perp \mathcal{N}^\infty$  which satisfies (1) with  $\epsilon$  multiplied by 2 and  $\tilde{v}_t^s = 0$  for  $s \geq t$ .

Thus, assume that  $v_s^s = 0$  for  $s > t$  and let  $\bar{\epsilon} > 0$  and  $\bar{x} \in \mathcal{N}^\infty$  be such that  $\tilde{\phi}(0) \geq Eh(\bar{x}) - \epsilon$ . Combined with (1) and noting that  $\langle \bar{x}, v \rangle = 0$ , we get

$$Eh(z) \geq Eh(\bar{x}) + \langle z - \bar{x}, v \rangle - \epsilon - \bar{\epsilon} \quad \forall z \in L^\infty.$$

Let  $z \in \text{dom } Eh \cap L^\infty$  and let  $\hat{z}$  be as in Assumption 1. By Theorem 14 in the appendix,

$$Eh(z) \geq Eh(\bar{x}) + \langle z - \bar{x}, v^a \rangle - \epsilon - \bar{\epsilon}, \quad (2)$$

and

$$0 \geq \langle \hat{z} - \bar{x}, v^s \rangle - \epsilon - \bar{\epsilon}. \quad (3)$$

Since  $\hat{z}^t = E_t z^t$  and  $v_s^s = 0$  for  $s > t$  by assumption, (3) means that

$$0 \geq \sum_{s=0}^t \langle E_t z_s - \bar{x}_s, v_s^s \rangle - \epsilon - \bar{\epsilon}.$$

Each term in the sum can be written as  $\langle z_s - \bar{x}_s, E_t^* v_s^s \rangle$ , where  $E_t^*$  denotes the adjoint of  $E_t : L^\infty(\Omega, \mathcal{F}, P; \mathbb{R}^{n_t}) \rightarrow L^\infty(\Omega, \mathcal{F}, P; \mathbb{R}^{n_t})$ . Moreover, since  $v \perp \mathcal{N}^\infty$ , we have  $E_t^* v_t = 0$  so, in the last term,  $E_t^* v_t^s = -E_t^* v_t^a = -E_t v_t^a$ . Thus, combining (3) and (2) gives

$$Eh(z) \geq Eh(\bar{x}) + \langle z - \bar{x}, \tilde{v} \rangle - 2\epsilon - 2\bar{\epsilon},$$

where

$$\tilde{v}_s = \begin{cases} v_s + E_t^* v_s^s & \text{for } s < t, \\ v_s^a - E_t v_s^a & \text{for } s = t, \\ v_t & \text{for } s > t. \end{cases}$$

It is easily checked that we still have  $\tilde{v} \in \mathcal{N}^\perp$  but now  $\tilde{v}_s^s = 0$  for every  $s \geq t$  as desired. Since  $\bar{\epsilon} > 0$  was arbitrary and  $\langle \tilde{x}, \tilde{v} \rangle = 0$ , we see that  $\tilde{v}$  satisfies (1) with  $\epsilon$  multiplied by 2. This completes the proof since  $z \in \text{dom } Eh \cap L^\infty$  was arbitrary.  $\square$

The general idea of the above proof is from [19, Theorem 2] where the imposed assumptions guarantee the strong continuity of  $\tilde{\phi}$  at the origin, which in turn guarantees subdifferentiability. The following two results give more general conditions under which the subdifferentiability holds.

**Theorem 5.** *Let Assumption 2 hold. If  $Eh$  is strongly continuous at a point of  $\mathcal{N}^\infty$  relative to  $\text{aff dom } Eh \cap L^\infty$ , then  $\tilde{\phi}$  is strongly subdifferentiable at the origin.*

*Proof.* We may assume without loss of generality that there exist  $M, \epsilon > 0$  such that  $Eh(z) \leq M$  for all  $z \in \text{aff dom } Eh$  with  $\|z\| \leq \epsilon$ . It is straightforward to check that  $\text{dom } \tilde{\phi} = \mathcal{N}^\infty + \text{dom } Eh$  and  $\text{aff dom } \tilde{\phi} = \mathcal{N}^\infty + \text{aff dom } Eh$ . Assumption 2 implies that if  $z \in \text{aff dom } \tilde{\phi}$ , then  $z - x_z \in \text{aff dom } Eh$  for some  $x_z \in \mathcal{N}^\infty$  with  $\|z - x_z\| \leq \rho \|{}^a z - z\|$ . Indeed, each  $z \in \text{aff dom } \tilde{\phi}$  can be expressed as  $z = x + w$ , where  $x \in \mathcal{N}^\infty$  and  $w \in \text{aff dom } Eh$ , while Assumption 2 gives the existence of a  $\tilde{x}_z \in \text{aff dom } Eh$  such that  $\|\tilde{x}_z - w\| \leq \rho \|{}^a w - w\| = \rho \|{}^a z - z\|$ . Setting  $x_z := \tilde{x}_z + x$ , we have  $z - x_z = w - \tilde{x}_z \in \text{aff dom } Eh$  and  $\|z - x_z\| \leq \rho \|{}^a z - z\|$  as claimed.

Now, if  $z \in \text{aff dom } \tilde{\phi}$  is such that  $\|z\| \leq \epsilon/2\rho$ , then  $\|z - x_z\| \leq \epsilon$ , so  $\tilde{\phi}(z) \leq Eh(z - x_z) \leq M$ . Since  $\tilde{\phi}(0)$  is finite by assumption, this implies that  $\tilde{\phi}$  is strongly continuous and thus subdifferentiable on  $\text{aff dom } \tilde{\phi}$ ; see [15, Theorem 11]. By the Hahn–Banach theorem, relative subgradients on  $\text{aff dom } \tilde{\phi}$  can be extended to subgradients on  $L^\infty$ .  $\square$

If  $Eh$  is a closed proper and convex with  $\text{aff dom } Eh$  closed, then  $Eh$  is continuous on  $\text{rint}_s \text{ dom } Eh$ , the *relative strong interior* of  $\text{dom } Eh$  (recall that the relative interior of a set is defined as its interior with respect to its affine hull). Indeed,  $\text{aff dom } Eh$  is a Banach space whenever it is closed, and then  $Eh$  is strongly continuous relative to  $\text{rint}_s \text{ dom } Eh$ ; see e.g. [15, Corollary 8B].

The following result gives sufficient conditions for  $\text{aff dom } Eh$  to be strongly closed and  $\text{rint}_s \text{ dom } Eh$  to be nonempty. Its proof, contained in the appendix, is obtained by modifying the proof of [14, Theorem 2] which required that  $\text{aff dom } h = \mathbb{R}^n$  almost surely. Recall that the set-valued mappings  $\omega \mapsto \text{dom } h$  and  $\omega \mapsto \text{aff dom } h$  are measurable; see [22, Proposition 14.8 and Exercise 14.12].

**Theorem 6.** *Assume that the set*

$$\mathcal{D} = \{x \in L^\infty(\text{dom } h) \mid \exists r > 0 : \mathbb{B}(x, r) \cap \text{aff dom } h \subseteq \text{dom } h \text{ } P\text{-a.e.}\}$$

*is nonempty and contained in  $\text{dom } Eh$ . Then  $Eh : L^\infty \rightarrow \overline{\mathbb{R}}$  is closed proper and convex,  $\text{aff dom } Eh$  is closed and  $\text{rint}_s \text{ dom } Eh = \mathcal{D}$ . In particular,  $Eh$  is strongly continuous throughout  $\mathcal{D}$  relative to  $\text{aff dom } Eh \cap L^\infty$ .*

**Remark 1.** Under the assumptions Theorem 6,  $Eh$  is subdifferentiable throughout  $\mathcal{D}$ . Indeed, the construction of  $y$  in the proof shows that  $y \in \partial Eh(x)$ , since  $y \in \partial h(x)$  almost surely.

**Example 1.** The extension of the integrability condition of [14, Theorem 2] in Theorem 6 is needed, for example, in problems of the form

$$\begin{aligned} & \text{minimize} && Eh_0(x) && \text{over } x \in \mathcal{N}^\infty \\ & \text{subject to} && Ax = b && P\text{-a.s.}, \end{aligned}$$

where  $h_0$  is a convex normal integrand such that  $h_0(x, \cdot) \in L^1$  for every  $x \in \mathbb{R}^n$ ,  $A$  is a measurable matrix and  $b$  is a measurable vector of appropriate dimensions such that the problem is feasible. Indeed, this fits the general format of (SP) with

$$h(x, \omega) = \begin{cases} h_0(x, \omega) & \text{if } A(\omega)x = b(\omega), \\ +\infty & \text{otherwise,} \end{cases}$$

so that  $\text{aff dom } h = \text{dom } h$  and  $\mathcal{D} = \{x \in L^\infty \mid Ax = b \text{ } P\text{-a.s.}\} = \text{dom } Eh$ .

### 3 Calculating conjugates and subgradients

This section applies the results of the previous sections to calculate subdifferentials and conjugates of certain integral functionals and conditional expectations of normal integrands.

#### 3.1 Integral functionals on $\mathcal{N}^\infty$

Let  $f$  be a normal integrand and consider the associated integral functional  $Ef$  with respect to the pairing  $\langle \mathcal{N}^\infty, \mathcal{N}^1 \rangle$ . We assume throughout this section that  $\text{dom } Ef \cap \mathcal{N}^\infty \neq \emptyset$ .

If  $x \in \mathcal{N}^\infty$  and  $v \in L^1(\partial f(x))$ , then  $Ef(x') \geq Ef(x) + \langle x' - x, v \rangle$  for all  $x' \in \mathcal{N}^\infty$ , so

$${}^a L^1(\partial f(x)) \subseteq \partial Ef(x). \quad (4)$$

The following theorem gives sufficient conditions for this to hold as an equality. We will use the convention that the subdifferential of a function at a point is nonempty unless the function is finite at the point.

**Theorem 7.** Assume that  $x^* \in \mathcal{N}^1$  is such that the function  $\tilde{\phi}_{x^*} : L^\infty \rightarrow \overline{\mathbb{R}}$ ,

$$\tilde{\phi}_{x^*}(z) := \inf_{x \in \mathcal{N}^\infty} E[f(x+z) - (x+z) \cdot x^*]$$

is closed at the origin. Then

$$(Ef)^*(x^*) = \inf_{v \in \mathcal{N}^\perp} Ef^*(x^* + v).$$

If  $\tilde{\phi}_{x^*}$  is subdifferentiable at the origin, then the infimum is attained. If this holds for every  $x^* \in \partial Ef(x)$ , then

$$\partial Ef(x) = {}^a L^1(\partial f(x)).$$

*Proof.* To prove the conjugate formula, note first that  $(Ef)^*(x^*) = -\tilde{\phi}_{x^*}(0) = -\text{cl } \tilde{\phi}_{x^*}(0) = \inf_y \tilde{\phi}_{x^*}^*(y)$ . By the Fenchel inequality, we always have  $(Ef)^*(x^*) \leq Ef^*(x^* + v)$  for all  $v \in \mathcal{N}^\perp$ , so we may assume that  $\tilde{\phi}_{x^*}$  is proper. In this case we have the expression  $\tilde{\phi}_{x^*}^*(y) = Ef^*(x^* + y) + \delta_{\mathcal{N}^\perp}(y)$ ; see the proof of Lemma 3.

Assume now that  $\tilde{\phi}_{x^*}$  is subdifferentiable at the origin for  $x^* \in \partial Ef(x)$ . Then the infimum in the expression for  $(Ef)^*(x^*)$  is attained and  $Ef(x) + (Ef)^*(x^*) = \langle x, x^* \rangle$ , so there is a  $v \in \mathcal{N}^\perp$  such that  $E[f(x) + f^*(x^* + v)] = E[x \cdot (x^* + v)]$ , and thus  $x^* + v \in \partial f(x)$ . Clearly,  $x^* = {}^a(x^* + v)$ . Thus,  $\partial Ef(x) \supseteq {}^a L^1(\partial f(x))$  while the reverse inclusion is always valid by (4).  $\square$

Combining the previous theorem with the results of Section 2, we get global conditions when the subdifferential of  $Ef$  coincides with the optional projection of the subdifferential of  $Ef$  with respect to the pairing  $\langle L^\infty, L^1 \rangle$ .

**Corollary 8.** *Let  $f$  satisfy Assumptions 1 and 2. If  $Ef$  is strongly continuous at a point of  $\mathcal{N}^\infty$  relative to  $\text{aff dom } f \cap L^\infty$ , then*

$$(Ef)^*(x^*) = \inf_{v \in \mathcal{N}^\perp} Ef^*(x^* + v) \quad \forall x^* \in \mathcal{N}^1$$

where the infimum is attained, and

$$\partial Ef(x) = {}^a L^1(\partial f(x)).$$

*Proof.* Let  $x^* \in \mathcal{N}^1$ . Since  $\text{dom } Ef \cap \mathcal{N}^\infty \neq \emptyset$ , we have  $\tilde{\phi}_{x^*}(0) < \infty$ . If  $\tilde{\phi}_{x^*}(0) = -\infty$ , then  $\tilde{\phi}_{x^*}$  is trivially closed at the origin. Assume now that  $\tilde{\phi}_{x^*}(0) > -\infty$ . The assumed properties of  $f$  imply that Assumptions 1 and 2 are satisfied by  $h(x, \omega) := f(x, \omega) - x \cdot x^*(\omega)$  and that  $Uh$  is continuous at a point of  $\mathcal{N}^\infty$  relative to  $\text{aff dom } fh \cap L^\infty$ . By Theorem 5 and Theorem 4,  $\tilde{\phi}_{x^*}$  is subdifferentiable at the origin. If  $x^* \in \partial(Ef)(x)$ , Fenchel's inequality  $\tilde{\phi}_{x^*}(0) \geq E[f(x) - x \cdot x^*] \geq -(Ef)^*(x^*)$  implies  $\tilde{\phi}_{x^*}(0) > -\infty$ . The assumptions of Theorem 7 are thus satisfied.  $\square$

Without the assumptions of Corollary 8, inclusion (4) may be strict. A simple example is given on page 176 of [21].

**Remark 2.** By Theorem 6, the continuity assumption in Corollary 8 holds, in particular, if

$$\mathcal{D} = \{x \in L^\infty(\text{dom } f) \mid \exists r > 0 : \mathbb{B}(x, r) \cap \text{aff dom } f \subseteq \text{dom } h \text{ P-a.e.}\}$$

is nonempty and contained in  $\text{dom } Ef$ .



### 3.2 Conditional expectation of a normal integrand

Results of the previous section allow for a simple proof of the interchange rule for subdifferentiation and conditional expectation of a normal integrand. Commutation of the two operations has been extensively studied ever since the introduction of the notion of a conditional expectation of a normal integrand in Bismut [3]; see Rockafellar and Wets [21], Truffert [23] and the references there in. The results of the previous section allow us to relax some of the continuity assumption made in earlier works.

Given a sub-sigma-algebra  $\mathcal{G} \subseteq \mathcal{F}$ , the  $\mathcal{G}$ -conditional expectation of a normal integrand  $f$  is a  $\mathcal{G}$ -measurable normal integrand  $E^{\mathcal{G}}f$  such that

$$(E^{\mathcal{G}}f)(x(\omega), \omega) = E^{\mathcal{G}}[f(x(\cdot), \cdot)](\omega) \quad P\text{-a.s.}$$

for all  $x \in L^0(\Omega, \mathcal{G}, P; \mathbb{R}^n)$  such that either  $f(x)^+ \in L^1$  or  $f(x)^- \in L^1$ . If  $\text{dom } Ef^* \cap L^1(\mathcal{F}) \neq \emptyset$ , then the conditional expectation exists and is unique in the sense that if  $\tilde{f}$  is another function with the above property, then  $\tilde{f}(\cdot, \omega) = (E^{\mathcal{G}}f)(\cdot, \omega)$  almost surely; see e.g. [23, Theorem 2.1.2].

The  $\mathcal{G}$ -conditional expectation of an  $\mathcal{F}$ -measurable set-valued mapping  $S : \Omega \rightrightarrows \mathbb{R}^n$  is a  $\mathcal{G}$ -measurable closed-valued mapping  $E^{\mathcal{G}}S$  such that

$$L^1(\mathcal{G}, E^{\mathcal{G}}S) = \text{cl}\{E^{\mathcal{G}}v \mid v \in L^1(\mathcal{F}, S)\}.$$

The conditional expectation is well-defined and unique as soon as  $S$  admits at least one integrable selection; see Hiai and Umegaki [7, Theorem 5.1].

The general form of “Jensen’s inequality” in the following lemma is from [23, Corollary 2.1.2]. We give a direct proof for completeness.

**Lemma 9.** *If  $f$  is a convex normal integrand such that  $\text{dom } Ef \cap L^\infty(\mathcal{G}) \neq \emptyset$  and  $\text{dom } Ef^* \cap L^1(\mathcal{F}) \neq \emptyset$ , then*

$$(E^{\mathcal{G}}f)^*(E^{\mathcal{G}}v) \leq E^{\mathcal{G}}f^*(v)$$

*almost surely for all  $v \in L^1(\mathcal{F})$  and*

$$\partial[E^{\mathcal{G}}f](x) \supseteq E^{\mathcal{G}}\partial f(x)$$

*for every  $x \in \text{dom } Ef \cap L^0(\mathcal{G})$ .*

*Proof.* Fenchel’s inequality  $f^*(v) \geq x \cdot v - f(x)$  and the assumption  $\text{dom } Ef \cap L^\infty(\mathcal{G}) \neq \emptyset$  imply that  $E^{\mathcal{G}}f^*(v)$  is well defined for all  $v \in L^1(\mathcal{F})$ . To prove the first claim, assume, for contradiction, that there is a  $v \in L^1(\mathcal{F})$  and a set  $A \in \mathcal{G}$  with  $P(A) > 0$  on which the inequality is violated. Passing to a subset of  $A$  if necessary, we may assume that  $E[\mathbb{1}_A E^{\mathcal{G}}f^*(v)] < \infty$  and thus,

$$E[\mathbb{1}_A (E^{\mathcal{G}}f)^*(E^{\mathcal{G}}v)] > E[\mathbb{1}_A E^{\mathcal{G}}f^*(v)] = E[\mathbb{1}_A f^*(v)].$$

This cannot happen since, by Fenchel’s inequality

$$E[\mathbb{1}_A f^*(v)] \geq \sup_{x \in L^\infty(\mathcal{G})} E[\mathbb{1}_A [x \cdot E^{\mathcal{G}}v - (E^{\mathcal{G}}f)(x)]] = E[\mathbb{1}_A (E^{\mathcal{G}}f)^*(E^{\mathcal{G}}v)],$$

where the equality follows by applying the interchange rule in  $L^\infty(A, \mathcal{G}, P; \mathbb{R}^n)$ .

Given  $v \in L^1(\mathcal{F}, \partial f(x))$ , we have

$$f(x) + f^*(v) = x \cdot v$$

almost surely. Let  $A^\nu = \{\|x\| \leq \nu\}$  so that  $\mathbb{1}_{A^\nu}x$  is bounded. Since  $\text{dom } Ef^* \cap L^1(\mathcal{F}) \neq \emptyset$ , Fenchel inequality implies that  $\mathbb{1}_{A^\nu}f(x)$  integrable. Taking conditional expectations,

$$\mathbb{1}_{A^\nu}E^\mathcal{G}f(x) + \mathbb{1}_{A^\nu}E^\mathcal{G}f^*(v) = \mathbb{1}_{A^\nu}x \cdot E^\mathcal{G}v,$$

so by the first part,

$$\mathbb{1}_{A^\nu}(E^\mathcal{G}f)(x) + \mathbb{1}_{A^\nu}(E^\mathcal{G}f)^*(E^\mathcal{G}v) \leq \mathbb{1}_{A^\nu}x \cdot E^\mathcal{G}v,$$

which means that  $E^\mathcal{G}v \in \partial(E^\mathcal{G}f)(x)$  almost surely on  $A^\nu$ . This finishes the proof since  $\nu$  was arbitrary.  $\square$

**Remark 3.** If in Lemma 9,  $f$  is normal  $\mathcal{G}$ -integrand, then the inequality can be written in the more familiar form  $f^*(E^\mathcal{G}v) \leq E^\mathcal{G}f^*(v)$ .

The following gives conditions for the equalities in Lemma 9 to hold.

**Theorem 10.** *Let  $f$  be a convex normal integrand such that  $\text{dom } Ef \cap L^\infty(\mathcal{G}) \neq \emptyset$  and  $\text{dom } Ef^* \cap L^1(\mathcal{F}) \neq \emptyset$ . If  $x^* \in L^1(\mathcal{G})$  is such that the function  $\tilde{\phi} : L^\infty \rightarrow \overline{\mathbb{R}}$ ,*

$$\tilde{\phi}(z) = \inf_{x \in L^\infty(\mathcal{G})} E[f(x+z) - (x+z) \cdot x^*]$$

*is subdifferentiable at the origin, then there is a  $v \in L^1(\mathcal{F})$  such that  $E^\mathcal{G}v = 0$  and*

$$(E^\mathcal{G}f)^*(x^*) = E^\mathcal{G}f^*(x^* + v).$$

*If  $x \in \text{dom } Ef \cap L^0(\mathcal{G})$  and the above holds for every  $x^* \in L^1(\mathcal{G}; \partial E^\mathcal{G}f(x))$ , then*

$$\partial[E^\mathcal{G}f](x) = E^\mathcal{G}\partial f(x).$$

*Proof.* Applying Theorem 7 with  $T = 0$  and  $\mathcal{F}_0 = \mathcal{G}$  gives the existence of a  $v \in L^1$  such that  $E^\mathcal{G}v = 0$  and

$$(Ef)^*(x^*) = Ef^*(x^* + v).$$

On the other hand,  $Ef = E(E^\mathcal{G}f)$  by definition, so  $(Ef)^*(x^*) = E(E^\mathcal{G}f)^*(x^*)$ , by [12, Theorem 2]. The first claim now follows from the fact that  $E^\mathcal{G}f^*(x^* + v) \geq (E^\mathcal{G}f)^*(x^*)$  almost surely, by Lemma 9.

If  $x^* \in L^1(\mathcal{G}; \partial E^\mathcal{G}f(x))$ , we have

$$(E^\mathcal{G}f)(x) + (E^\mathcal{G}f)^*(x^*) = x \cdot x^* \quad P\text{-a.s.}$$

By the first part, there is a  $v \in L^1(\mathcal{F})$  such that  $E^\mathcal{G}v = 0$  and

$$(E^\mathcal{G}f)(x) + E^\mathcal{G}f^*(x^* + v) = x \cdot x^* \quad P\text{-a.s.}$$

It follows that

$$E[f(x) + f^*(x^* + v) - x \cdot (x^* + v)] = 0,$$

which by the Fenchel inequality, implies  $x^* + v \in \partial f(x)$  so  $\partial[E^\mathcal{G}f](x) \subseteq E^\mathcal{G}\partial f(x)$ . Combining this with Lemma 9 completes the proof.  $\square$

Sufficient conditions for the subdifferentiability condition are again obtained from Theorems 5 and 6.

**Corollary 11.** *Let  $f$  be a convex normal integrand such that  $\text{dom } Ef^* \cap L^1(\mathcal{F}) \neq \emptyset$ ,  $E^\mathcal{G}x \in \text{dom } Ef$  for all  $x \in \text{dom } Ef \cap L^\infty$  and  $Ef$  is strongly continuous at a point of  $L^\infty(\mathcal{G})$  relative to  $\text{aff dom } Ef \cap L^\infty$ . Then for every  $x^* \in L^1(\mathcal{G})$  there is a  $v \in L^1(\mathcal{F})$  such that  $E^\mathcal{G}v = 0$  and*

$$(E^\mathcal{G}f)^*(x^*) = E^\mathcal{G}f^*(x^* + v).$$

Moreover,

$$\partial[E^\mathcal{G}f](x) = E^\mathcal{G}\partial f(x).$$

for every  $x \in \text{dom } Ef \cap L^0(\mathcal{G})$ .

*Proof.* Analogously to Corollary 8, the additional conditions guarantee the subdifferentiability condition in Theorem 10; see the remarks after Assumption 2.  $\square$

The above subdifferential formula was obtained in [21] while the expression for the conjugate was given in [23, Corollary 2.2.3]. Both assumed the stronger condition that  $Ef$  be continuous at a point  $x \in L^\infty(\mathcal{G})$  relative to all of  $L^\infty(\mathcal{F})$ . A more abstract condition (not requiring the relative continuity assumed here) for the subdifferential formula is given in the corollary in Section 2.2.2 of [23].

Let  $g$  be a convex normal integrand. The  $\mathcal{G}$ -conditional expectation of the epigraphical mapping  $\text{epi } g$  is also an epigraphical mapping of some normal integrand as soon as  $\text{epi } g$  has an integrable selection; see [23, p. 136 and 140]. We denote by  ${}^\mathcal{G}g$  the normal integrand whose epigraphical mapping is the  $\mathcal{G}$ -conditional expectation of the epigraphical mapping of  $g$ . We get from [23, Theorem 2.1.2 and Corollary 2.1.1.1] that

$$({}^\mathcal{G}g)^* = E^\mathcal{G}(g^*) \tag{5}$$

whenever there exists  $y \in \text{dom } Eg \cap L^1$  and  $x \in \text{dom } Eg^* \cap L^0(\mathcal{G})$ . Thus results of this section concerning with  $(E^\mathcal{G}f)^*$  can be expressed as well in terms  ${}^\mathcal{G}(f^*)$ .

### 3.3 Dynamic programming

Consider again problem (SP) and define extended real-valued functions  $h_t, \tilde{h}_t : \mathbb{R}^{n_1 + \dots + n_t} \times \Omega \rightarrow \overline{\mathbb{R}}$  by the recursion

$$\begin{aligned} \tilde{h}_T &= h, \\ h_t &= E_t \tilde{h}_t, \\ \tilde{h}_{t-1}(x^{t-1}, \omega) &= \inf_{x_t \in \mathbb{R}^{n_t}} h_t(x^{t-1}, x_t, \omega). \end{aligned} \tag{6}$$

This far reaching generalization of the classical dynamic programming recursion for control systems was introduced in [19] and [6]. The following result from [10] relaxes the compactness assumptions made in [19] and [6]. In the context of financial mathematics, this allows for various extensions of certain fundamental results in financial mathematics; see [10] for details.

**Theorem 12** ([10]). *Assume that  $h \geq m$  for an  $m \in L^1$  and that*

$$\{x \in \mathcal{N} \mid h^\infty(x) \leq 0 \text{ } P\text{-a.s.}\}$$

*is a linear space. The functions  $h_t$  are then well-defined normal integrands and we have for every  $x \in \mathcal{N}$  that*

$$Eh_t(x^t) \geq \phi(0) \quad t = 0, \dots, T. \quad (7)$$

*Optimal solutions  $x \in \mathcal{N}$  exist and they are characterized by the condition*

$$x_t(\omega) \in \underset{x_t}{\operatorname{argmin}} h_t(x^{t-1}(\omega), x_t, \omega) \quad P\text{-a.s.} \quad t = 0, \dots, T.$$

*which is equivalent to having equalities in (7).*

Consider now the dual problem

$$\text{minimize} \quad Eh^*(v) \quad \text{over } v \in \mathcal{N}^\perp$$

from Theorem 1. We know that the optimum dual value is at least  $-\phi(0)$  and that if the values are equal, the shadow prices of information are exactly the dual solutions. Note also that when the functions  $h_t$  and  $\tilde{h}_t$  in the dynamic programming equations are well-defined, their conjugates solve the *dual dynamic programming* equations

$$\begin{aligned} \tilde{g}_T &= h^*, \\ g_t &= \mathcal{F}_t \tilde{g}_t, \\ \tilde{g}_{t-1}(v^{t-1}, \omega) &= g_t(v^{t-1}, 0, \omega). \end{aligned} \quad (8)$$

Much like Theorem 12 characterizes optimal primal solutions in terms of the dynamic programming equations (6), the following result characterizes optimal dual solutions in terms of the dual recursion (8).

**Theorem 13.** *Assume that the dual problem is proper and that there is a feasible  $\bar{x} \in \mathcal{N}^\infty$  for the primal problem. Then the dual dynamic programming equations are well-defined and we have for every  $v \in \mathcal{N}^\perp$  that*

$$Eg_t(E_tv^t) \geq -\phi(0) \quad t = 0, \dots, T. \quad (9)$$

*In the absence of a duality gap, optimal dual solutions are characterized by having equalities in (9) while  $x \in \mathcal{N}$  and  $v \in \mathcal{N}^\perp$  are primal and dual optimal, respectively, if and only if  $Eg(x) < \infty$ ,  $Eg^*(v) < \infty$  and*

$$Eg_t^*(x^t) + Eg_t(E_tv^t) = 0 \quad t = 0, \dots, T,$$

which is equivalent to having

$$E_t v^t \in \partial g_t(x^t) \quad P\text{-a.s.} \quad t = 0, \dots, T.$$

*Proof.* Let  $\bar{v} \in \mathcal{N}^\perp$  be feasible for the dual problem. We first show inductively that  $E_{t+1} \bar{v}^t \in \text{dom } E \tilde{g}_t$  and  $\bar{x}^t \in \text{dom } E \tilde{g}_t^*$  which implies, in particular, that each  $g_t = \mathcal{F}_t \tilde{g}_t$  is well-defined. For  $t = T$ , this is trivial. Assume that the claim holds for some  $t \leq T$ . Then, for every  $v \in \mathcal{N}^\perp$ , we have

$$\tilde{g}_{t-1}(E_t v^{t-1}) = g_t(E_t v^t) \leq E_t \tilde{g}_t(E_{t+1} v^t) = E_t g_{t+1}(E_{t+1} v^{t+1}), \quad (10)$$

where the inequality follows from the induction hypotheses  $\bar{x}^t \in \text{dom } E \tilde{g}_t^*$  and Lemma 9. Thus  $E_t \bar{v}^{t-1} \in \text{dom } E \tilde{g}_{t-1}$ . By definition,  $\tilde{g}_{t-1}(v^{t-1}) = g_t(v^{t-1}, 0)$ , so  $\tilde{g}_{t-1}^*(x^{t-1}, \omega) = \text{cl inf}_{x_t} g_t^*(x^{t-1}, x_t, \omega)$ . By (5),  $g_t^* = E^{\mathcal{F}_t}(\tilde{g}_t^*)$ . Thus, for every  $x \in \mathcal{N} \cap \text{dom } E g$ , we have

$$\tilde{g}_{t-1}^*(x^{t-1}) \leq g_t^*(x^t) \leq E_t g_{t+1}^*(x^{t+1}). \quad (11)$$

Thus  $\bar{x}^{t-1} \in \text{dom } E \tilde{g}_{t-1}^*$  which finishes the induction proof.

Let  $x \in \text{dom } E g \cap \mathcal{N}$ , and  $v \in \text{dom } E g^* \cap \mathcal{N}^\perp$ . Combining (10) and (11) with the fact that  $g_0^*(x_0) \geq -g_0(0)$  gives

$$E g(v) \geq E g_t(E_t v^t) \geq E g_0(0) \geq -E g_0^*(x_0) \geq -E g_t^*(x^t) \geq -E g^*(x) \quad (12)$$

for all  $t$ . In particular, (9) holds. In the absence of duality gap, (12) also imply that optimal dual solutions are characterized by having inequalities in (9). Likewise, we get from (12) that  $x$  and  $v$  are primal and dual optimal, respectively, if and only if

$$E g_t^*(x^t) + E g_t(E_t v^t) = 0 \quad t = 0, \dots, T.$$

By Fenchel's inequality,  $g_t^*(x^t) + g_t(E_t v^t) \geq x^t \cdot (E_t v^t)$ , so, by [11, Lemma 1],  $E[x^t \cdot (E_t v^t)] = 0$  whenever the left side is integrable. Thus  $E g_t^*(x^t) + E g_t(E_t v^t) = 0$  is equivalent to having  $g_t^*(x^t) + g_t(E_t v^t) = x^t \cdot (E_t v^t)$  almost surely, which means that

$$E_t v^t \in \partial g_t(x^t)$$

almost surely. □

## 4 Appendix

This appendix contains the proofs of Theorems 1 and 6 as well as Theorem 14 below which was used in the proof of Theorem 4. Both Theorem 14 and 6 are simple refinements of well-known results on convex integral functionals, both originally due to Terry Rockafellar.

**Theorem 14.** *Let  $h$  be a convex normal integrand and  $\bar{z} \in \text{dom } E h \cap L^\infty$ . If  $v \in (L^\infty)^*$  and  $\epsilon \geq 0$  such that*

$$E h(z) \geq E h(\bar{z}) + \langle z - \bar{z}, v \rangle - \epsilon \quad \forall z \in L^\infty, \quad (13)$$

then

$$Eh(z) \geq Eh(\bar{z}) + \langle z - \bar{z}, v^a \rangle - \epsilon \quad \forall z \in L^\infty,$$

and

$$0 \geq \langle z - \bar{z}, v^s \rangle - \epsilon \quad \forall z \in \text{dom } Eh.$$

*Proof.* Let  $z \in \text{dom } Eh \cap L^\infty$  and define  $z^\nu := \mathbb{1}_{A^\nu} \bar{z} + \mathbb{1}_{\Omega \setminus A^\nu} z$  where  $A^\nu$  are the sets in the characterization of the singular component  $v^s$ . We have  $h(z^\nu) \rightarrow h(z)$  almost surely and  $z^\nu \rightarrow z$  both weakly and almost surely. Thus, since  $h(z^\nu) \leq \max\{h(\bar{z}), h(z)\}$ , Fatou's lemma and (13) give,

$$\begin{aligned} Eh(z) &\geq \limsup Eh(z^\nu) \geq Eh(\bar{z}) + \limsup \langle z^\nu - \bar{z}, v \rangle - \epsilon \\ &= Eh(\bar{z}) + \langle z - \bar{z}, v^a \rangle - \epsilon, \end{aligned}$$

where the equality holds since  $z^\nu - \bar{z} = \mathbb{1}_{\Omega \setminus A^\nu} (z - \bar{z})$ , so that

$$\langle z^\nu - \bar{z}, v \rangle = \langle z^\nu - \bar{z}, v^a \rangle \rightarrow \langle z - \bar{z}, v^a \rangle.$$

Now let  $z^\nu := \mathbb{1}_{A^\nu} z + \mathbb{1}_{\Omega \setminus A^\nu} \bar{z}$ . We have that  $h(z^\nu) \rightarrow h(\bar{z})$  almost surely and  $z^\nu \rightarrow \bar{z}$  both weakly and almost surely. Since  $h(z^\nu) \leq \max\{h(z), h(\bar{z})\}$ , Fatou's lemma and (13) give,

$$\begin{aligned} Eh(\bar{z}) &\geq \limsup Eh(z^\nu) \geq Eh(\bar{z}) + \limsup \langle z^\nu - \bar{z}, v \rangle - \epsilon \\ &= Eh(\bar{z}) + \langle z - \bar{z}, v^s \rangle - \epsilon, \end{aligned}$$

where the equality holds since  $z^\nu - \bar{z} = \mathbb{1}_{A^\nu} (z - \bar{z})$  so that

$$\langle z^\nu - \bar{z}, v \rangle = \langle z^\nu - \bar{z}, v^a \rangle + \langle z^\nu - \bar{z}, v^s \rangle \rightarrow \langle z - \bar{z}, v^s \rangle$$

which completes the proof.  $\square$

*Proof of Theorem 1.* Let  $D := \{x \in \mathcal{N} \mid \exists z \in L^\infty : Eh(x + z) < \infty\}$ . By the interchange rule [22, Theorem 14.60],

$$\begin{aligned} \phi^*(v) &= \sup_{z \in L^\infty} \{\langle z, v \rangle - \phi(z)\} \\ &= \sup_{x \in \mathcal{N}} \sup_{z \in L^\infty} E[z \cdot v - h(x + z)] \\ &= \sup_{x \in D} E[x \cdot v - h^*(v)]. \end{aligned}$$

Since  $\text{dom } Eh \cap L^\infty \neq \emptyset$  implies  $\mathcal{N}^\infty \subseteq D$ , we have  $\phi^*(v) = +\infty$  for  $v \notin \mathcal{N}^\perp$ . By the Fenchel inequality,  $h(x) + h^*(v) \geq x \cdot v$  for all  $x, v \in \mathbb{R}^n$ , so [11, Lemma 1] implies  $E(x \cdot v) = 0$  for every  $x \in D$  and  $v \in \mathcal{N}^\perp \cap \text{dom } Eh^*$ . The second claim follows from the first one by noting that  $v \in \partial\phi(0)$  if and only if  $-\phi^*(v) = \phi(0)$ . Finally,  $x \in \mathcal{N}$  and  $v \in \mathcal{N}^\perp$  are optimal with  $Eh(x) + Eh^*(v) = 0$ , if and only if  $Eh(x) < \infty$ ,  $Eh^*(v) < \infty$  and the above Fenchel inequality holds almost surely as an equality, or equivalently,  $v \in \partial h(x)$  almost surely.  $\square$

*Proof of Theorem 6.* Translating, if necessary, we may assume  $0 \in \mathcal{D}$  so that  $L^\infty(\text{aff dom } h) \subseteq \cup_{\lambda > 0} \lambda \mathcal{D} \subseteq \text{aff } \mathcal{D}$ . By assumption,  $\mathcal{D} \subseteq \text{dom } Eh \cap L^\infty \subseteq L^\infty(\text{dom } h) \subseteq L^\infty(\text{aff dom } h)$ . Thus,  $\text{aff } \mathcal{D} = \text{aff}(\text{dom } Eh \cap L^\infty) = \text{aff } L^\infty(\text{dom } h) = L^\infty(\text{aff dom } h)$  which is a closed set. The above also implies  $\text{rint}_s \mathcal{D} \subseteq \text{rint}_s \text{dom } Eh \subseteq \text{rint}_s L^\infty(\text{dom } h)$ . Clearly  $\text{rint}_s L^\infty(\text{dom } h) \subseteq \mathcal{D}$  while  $\text{rint}_s \mathcal{D} = \mathcal{D}$ . It remains to prove that  $Eh$  is closed and proper.

Let  $\bar{r} > 0$  be such that  $\mathbb{B}(0, \bar{r}) \cap \text{aff dom } h \subseteq \text{rint}_s \text{dom } h$  almost surely and let  $\pi(\omega)$  be the projection from  $\mathbb{R}^d$  to  $\text{aff dom } h(\cdot, \omega)$ . There exist  $x^i \in \mathbb{R}^d$ ,  $i = 0, \dots, d$  and  $r > 0$  such that  $|x^i| < \bar{r}$  and  $\mathbb{B}(0, r)$  belongs to the interior of the convex hull of  $\{x^i \mid i = 0, \dots, d\}$ . By [22, Exercise 14.17],  $\pi x$  is measurable for every measurable  $x$ , so each  $\pi x^i$  belongs to  $\mathcal{D}$  and thus,

$$\alpha := \max_{i=0, \dots, d} h(\pi x^i)$$

is integrable. Since  $0 \in \text{rint dom } h$  almost surely, the closed convex-valued mapping

$$\Gamma(\omega) = \partial h(0, \omega) \cap \text{aff dom } h(\cdot, \omega)$$

is nonempty-valued and measurable. Indeed, the measurability follows from [22, Proposition 14.11 and Theorem 14.56], and nonemptiness follows from [13, Theorem 23.4] and the simple fact that  $\pi(\partial h) \subseteq \partial h$ . By [22, Corollary 14.6], there exists  $y \in L^0(\Gamma)$ . By the definition of subdifferential,

$$y(\omega) \cdot x \leq h(x, \omega) - h(0, \omega)$$

for all  $x \in \mathbb{R}^d$ , and, in particular,  $h^*(y) \leq -h(0)$ . Therefore,

$$\begin{aligned} r|y(\omega)| &= \sup_{x \in \mathbb{B}(0, r)} \{y(\omega) \cdot x\} \\ &= \sup_{x \in \mathbb{B}(0, r)} \{y(\omega) \cdot \pi(\omega)x\} \\ &\leq \sup_{x \in \mathbb{B}(0, r)} h(\pi(\omega)x, \omega) - h(0, \omega) \\ &\leq \alpha(\omega) - h(0, \omega), \end{aligned}$$

where the second equality holds since  $y(\omega) \in \text{aff dom } h(\cdot, \omega)$  almost surely. Thus,  $y \in L^1$  and  $y \in \text{dom } Eh^*$  so, by [12, Theorem 2],  $Eh$  is closed and proper.  $\square$

## References

- [1] K. Back and S. R. Pliska. The shadow price of information in continuous time decision problems. *Stochastics*, 22(2):151–186, 1987.
- [2] S. Biagini, T. Pennanen, and A.-P. Perkkiö. Duality and optimality conditions in stochastic optimization and mathematical finance. *manuscript*, 2015.

- [3] J.-M. Bismut. Intégrales convexes et probabilités. *J. Math. Anal. Appl.*, 42:639–673, 1973.
- [4] M. H. A. Davis. Dynamic optimization: a grand unification. In *Proceedings of the 31st IEEE Conference on Decision and Control*, volume 2, pages 2035 – 2036, 1992.
- [5] M. H. A. Davis and G. Burstein. A deterministic approach to stochastic optimal control with application to anticipative control. *Stochastics and Stochastics Reports*, 40(3&4):203–256, 1992.
- [6] I. V. Evstigneev. Measurable selection and dynamic programming. *Math. Oper. Res.*, 1(3):267–272, 1976.
- [7] F. Hiai and H. Umegaki. Integrals, conditional expectations, and martingales of multivalued functions. *J. Multivariate Anal.*, 7(1):149–182, 1977.
- [8] T. Pennanen. Convex duality in stochastic optimization and mathematical finance. *Mathematics of Operations Research*, 36(2):340–362, 2011.
- [9] T. Pennanen. Optimal investment and contingent claim valuation in illiquid markets. *Finance Stoch.*, 18(4):733–754, 2014.
- [10] T. Pennanen and A.-P. Perkkiö. Stochastic programs without duality gaps. *Mathematical Programming*, 136(1):91–110, 2012.
- [11] A.-P. Perkkiö. Stochastic programs without duality gaps for objectives without a lower bound. *manuscript*, 2014.
- [12] R. T. Rockafellar. Integrals which are convex functionals. *Pacific J. Math.*, 24:525–539, 1968.
- [13] R. T. Rockafellar. *Convex analysis*. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.
- [14] R. T. Rockafellar. Integrals which are convex functionals. II. *Pacific J. Math.*, 39:439–469, 1971.
- [15] R. T. Rockafellar. *Conjugate duality and optimization*. Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1974.
- [16] R. T. Rockafellar. On the equivalence of multistage recourse models in stochastic optimization. pages 314–321. *Lecture Notes in Econom. and Math. Systems*, Vol. 107, 1975.
- [17] R. T. Rockafellar and R. J.-B. Wets. Continuous versus measurable recourse in  $N$ -stage stochastic programming. *J. Math. Anal. Appl.*, 48:836–859, 1974.
- [18] R. T. Rockafellar and R. J.-B. Wets. Stochastic convex programming: Kuhn-Tucker conditions. *J. Math. Econom.*, 2(3):349–370, 1975.



- [19] R. T. Rockafellar and R. J.-B. Wets. Nonanticipativity and  $L^1$ -martingales in stochastic optimization problems. *Math. Programming Stud.*, (6):170–187, 1976. Stochastic systems: modeling, identification and optimization, II (Proc. Sympos., Univ Kentucky, Lexington, Ky., 1975).
- [20] R. T. Rockafellar and R. J.-B. Wets. The optimal recourse problem in discrete time:  $L^1$ -multipliers for inequality constraints. *SIAM J. Control Optimization*, 16(1):16–36, 1978.
- [21] R. T. Rockafellar and R. J.-B. Wets. On the interchange of subdifferentiation and conditional expectations for convex functionals. *Stochastics*, 7(3):173–182, 1982.
- [22] R. T. Rockafellar and R. J.-B. Wets. *Variational analysis*, volume 317 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1998.
- [23] A. Truffert. Conditional expectation of integrands and random sets. *Ann. Oper. Res.*, 30(1-4):117–156, 1991. Stochastic programming, Part I (Ann Arbor, MI, 1989).
- [24] R. J-B Wets. On the relation between stochastic and deterministic optimization. In A. Bensoussan and J.L. Lions, editors, *Control Theory, Numerical Methods and Computer Systems Modelling*, volume 107 of *Lecture Notes in Economics and Mathematical Systems*, pages 350–361. Springer, 1975.
- [25] K. Yosida and E. Hewitt. Finitely additive measures. *Trans. Amer. Math. Soc.*, 72:46–66, 1952.